EDP ELIPTICAS

ECUACION DE LAPLACE:

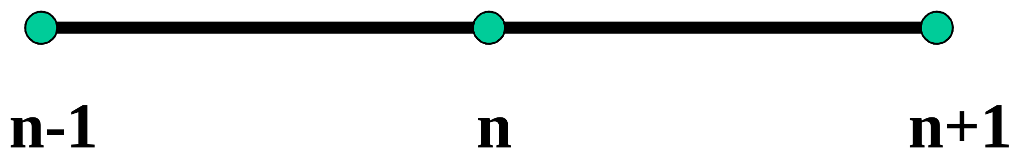
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

ECUACION DE POISSON:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

EDP ELIPTICAS
APROXIMACIONES DE LA SEGUNDA DERIVADA

$$f'' = \frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}$$



APLICANDO LA APROXIMACION A LA ECUACION DE LAPLACE

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 u = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

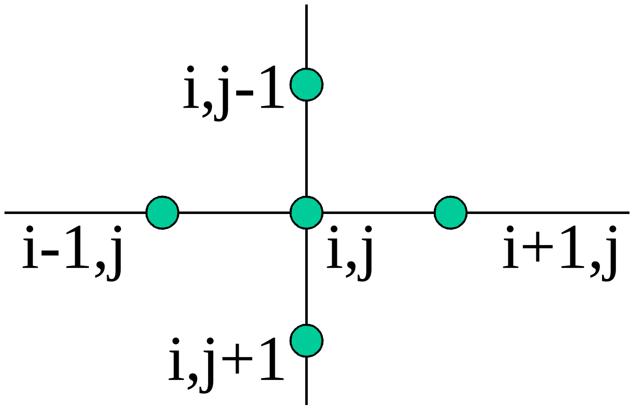
$$\Delta x = \Delta y = h$$

$$\nabla^2 u = \frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right)$$

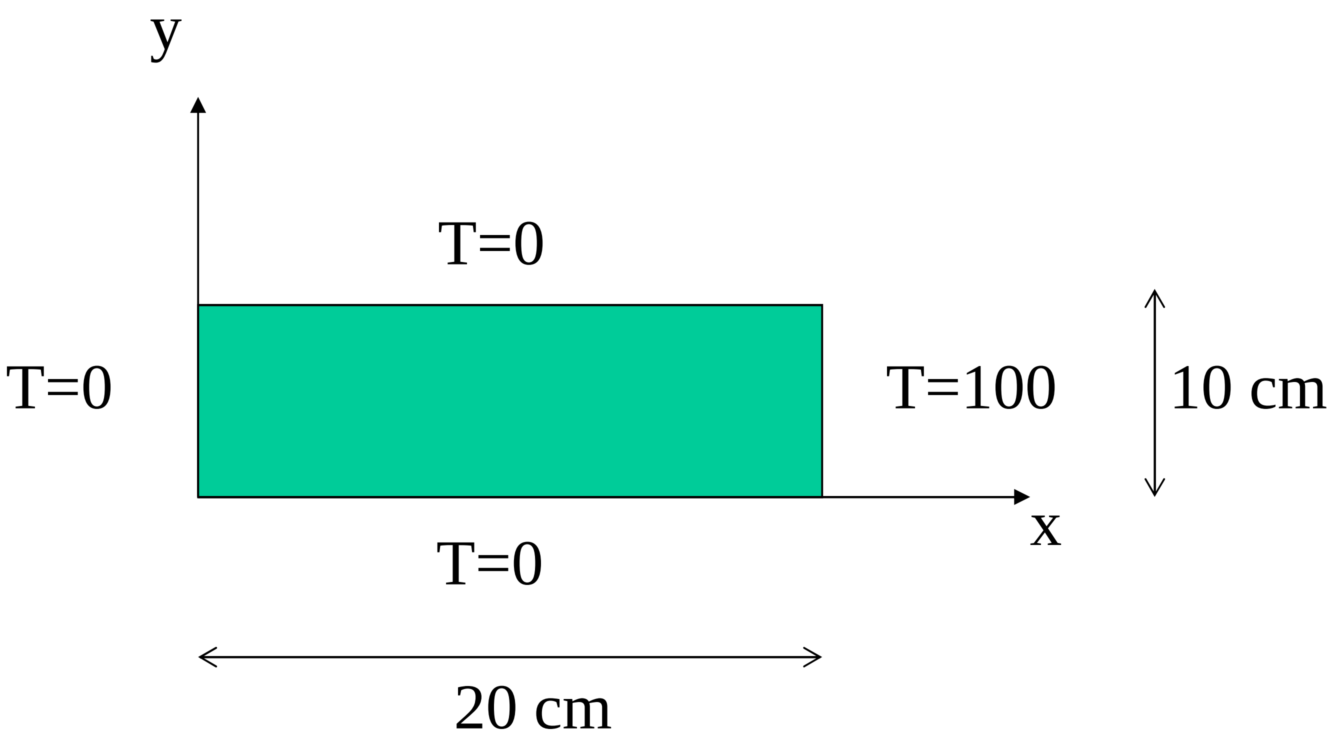
REPRESENTACION SIMBOLICA DE LA ECUACION DE LAPLACE

$$\nabla^2 u = \frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right)$$

$$\nabla^2 u = \frac{1}{h^2}\begin{Bmatrix} & +1 & \\ +1 & -4 & +1 \\ & +1 & \end{Bmatrix} u_{i,j} = 0$$



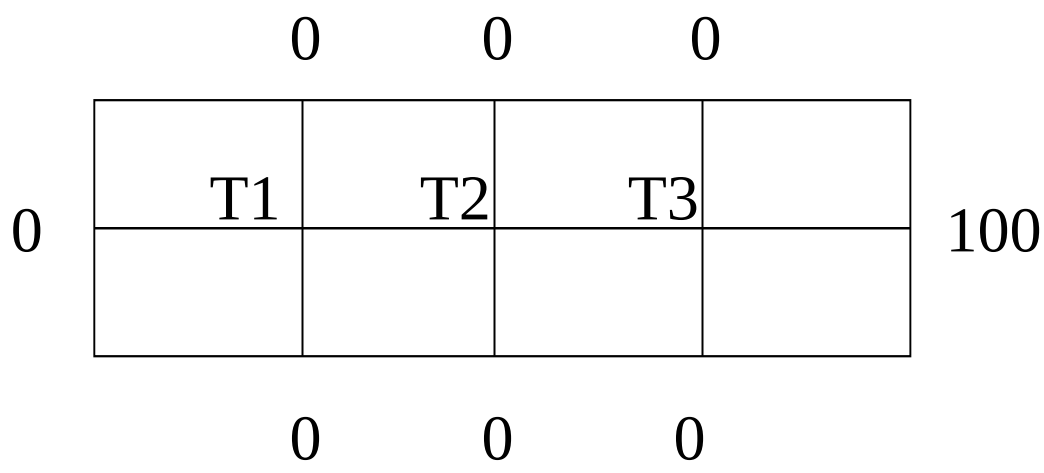
EJEMPLO DE
ECUACION DE LAPLACE



$$\nabla^2 u = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\nabla^2 T = \frac{1}{h^2}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}\right)$$

$$\begin{cases} \frac{1}{25}\left[T_2 + 0 + 0 + 0 - 4T_1\right] = 0 \\ \frac{1}{25}\left[T_3 + T_1 + 0 + 0 - 4T_2\right] = 0 \\ \frac{1}{25}\left[100 + T_2 + 0 + 0 - 4T_3\right] = 0 \end{cases}$$



$$\begin{cases} -4T_1 + T_2 = 0 \\ T_1 - 4T_2 + T_3 = 0 \\ T_2 - 4T_3 = -100 \end{cases}$$

Método de Liebmann:

$$\begin{cases} T_1 = \dfrac{T_2}{4} \\ T_2 = \dfrac{T_1 + T_3}{4} \\ T_3 = \dfrac{T_2 + 100}{4} \end{cases}$$

n	T1	T2	T3
0	2	8	30
1	2,0000	8,0000	27,0000
2	2,0000	7,2500	27,0000
3	1,8125	7,2500	26,8125
4	1,8125	7,1563	26,8125
5	1,7891	7,1563	26,7891
6	1,7891	7,1445	26,7891
7	1,7861	7,1445	26,7861
8	1,7861	7,1431	26,7861
9	1,7858	7,1431	26,7858
10	1,7858	7,1429	26,7858
11	1,7857	7,1429	26,7857
12	1,7857	7,1429	26,7857

OPERADOR LAPLACIANO EN TRES DIMENSIONES

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\nabla^2 u = \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} +$$

$$\frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} +$$

$$\frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2} = 0$$

$$\Delta x = \Delta y = \Delta z = h$$

$$\nabla^2 u = \frac{1}{h^2}\begin{pmatrix} u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} \\ + u_{i,j,k+1} + u_{i,j,k-1} - 6u_{i,j} \end{pmatrix}$$

$$\nabla^2 u = \frac{1}{h^2}\begin{Bmatrix} & +1 & +1 \\ +1 & -6 & +1 \\ +1 & +1 & \end{Bmatrix} u_{i,j} = 0$$

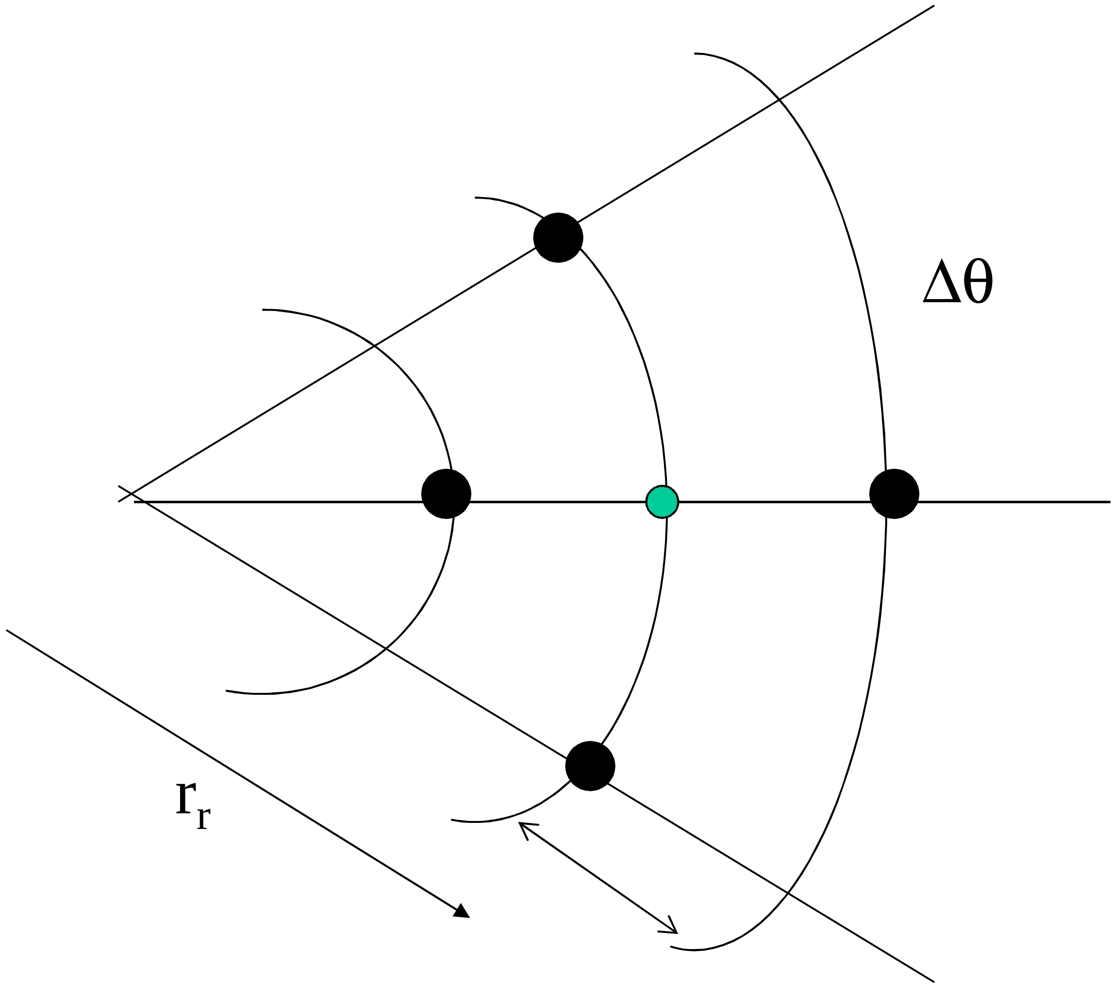
OPERADOR LAPLACIANO EN COORDENADAS POLARES

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\nabla^2 u = \frac{u_{r+1,\theta} - 2u_{r,\theta} + u_{r-1,\theta}}{(\Delta r)^2} +$$

$$\frac{1}{r_r}\frac{u_{r+1,\theta} - u_{r-1,\theta}}{2(\Delta r)} +$$

$$\frac{1}{r_r{}^2}\frac{u_{r,\theta+1} - 2u_{r,\theta} + u_{r,\theta-1}}{(\Delta\theta)^2} = 0$$


Δθ
r_r

EDP PARABOLICAS

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \alpha \frac{\partial u}{\partial t}$$

METODOS
- EXPLICITOS
- IMPLICITOS

METODOS EXPLICITOS

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_i^{j}}{\Delta t}$$

$$\frac{u_{i+1}^{j} - 2u_{i}^{j} + u_{i-1}^{j}}{(\Delta x)^{2}} = \alpha \frac{u_{i}^{j+1} - u_{i}^{j}}{\Delta t}$$

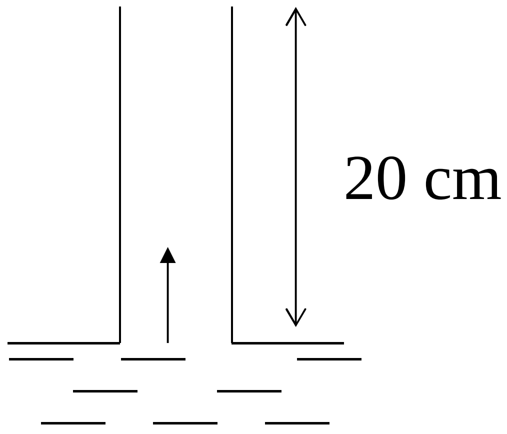
$$u_{i}^{j+1} = \frac{\Delta t}{\alpha(\Delta x)^{2}}\left(u_{i+1}^{j} + u_{i-1}^{j}\right) + \left(1 - \frac{2\Delta t}{\alpha(\Delta x)^{2}}\right)u_{i}^{j}$$

Si: $\frac{\Delta t}{\alpha(\Delta x)^{2}} = \frac{1}{2}$

$$u_{i}^{j+1} = \frac{1}{2}\left(u_{i+1}^{j} + u_{i-1}^{j}\right)$$

EJEMPLO DEL TUBO HUECO

Un tubo hueco de 20 cm de longitud está inicialmente lleno con aire conteniendo 2% de etilalcohol (vapor). Al fondo del tubo está una piscina de alcohol el cual se evapora en el gas estancado de arriba.



$$D\frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t}$$

$$C(x,0) = 2\%$$

$$C(0,t) = 0\%$$

$$C(20,t) = 10\%$$

$$\Delta x = 4\text{cm}$$

$$\frac{D\Delta t}{(\Delta x)^2} = \frac{1}{2}$$

$$\Delta t = 67{,}2\text{s}$$

$$C_i^{j+1} = \frac{1}{2}\left(C_{i+1}^{j} + C_{i-1}^{j}\right)$$

EJEMPLO DEL TUBO HUECO

		Concentración en alcohol				
Tiempo (s)	**x=0**	**x=4**	**x=8**	**x=12**	**x=16**	**x=20**
0	0,000	2,000	2,000	2,000	2,000	10,000
67,2	0,000	1,000	2,000	2,000	6,000	10,000
134,4	0,000	1,000	1,500	4,000	6,000	10,000
201,6	0,000	0,750	2,500	3,750	7,000	10,000
268,8	0,000	1,250	2,250	4,750	6,875	10,000
336	0,000	1,125	3,000	4,563	7,375	10,000
403,2	0,000	1,500	2,844	5,188	7,281	10,000
470,4	0,000	1,422	3,344	5,063	7,594	10,000
537,6	0,000	1,672	3,242	5,469	7,531	10,000
604,8	0,000	1,621	3,570	5,387	7,734	10,000
672	0,000	1,785	3,504	5,652	7,693	10,000
739,2	0,000	1,752	3,719	5,599	7,826	10,000
806,4	0,000	1,859	3,675	5,772	7,799	10,000

$$\Delta x = 4\text{cm}$$

$$\frac{D\Delta t}{(\Delta x)^2} = \frac{1}{4}$$

$$\Delta t = 33{,}6\text{s}$$

$$C_i^{j+1} = \frac{1}{4}\left(C_{i+1}^{j} + C_{i-1}^{j}\right) + \frac{1}{2}C_i^{j}$$

		Concentración en alcohol				
Tiempo (s)	**x=0**	**x=4**	**x=8**	**x=12**	**x=16**	**x=20**
0	0,000	2,000	2,000	2,000	2,000	10,000
33,6	0,000	1,500	2,000	2,000	4,000	10,000
67,2	0,000	1,250	1,875	2,500	5,000	10,000
100,8	0,000	1,094	1,875	2,969	5,625	10,000
134,4	0,000	1,016	1,953	3,359	6,055	10,000
168	0,000	0,996	2,070	3,682	6,367	10,000
201,6	0,000	1,016	2,205	3,950	6,604	10,000
235,2	0,000	1,059	2,344	4,177	6,790	10,000
268,8	0,000	1,115	2,481	4,372	6,939	10,000
302,4	0,000	1,178	2,612	4,541	7,063	10,000
336	0,000	1,242	2,736	4,689	7,167	10,000
369,6	0,000	1,305	2,851	4,820	7,256	10,000
403,2	0,000	1,365	2,957	4,937	7,333	10,000
436,8	0,000	1,422	3,054	5,041	7,401	10,000
470,4	0,000	1,474	3,143	5,134	7,460	10,000
504	0,000	1,523	3,223	5,218	7,514	10,000
537,6	0,000	1,567	3,297	5,293	7,561	10,000
571,2	0,000	1,608	3,363	5,361	7,604	10,000
604,8	0,000	1,645	3,424	5,422	7,642	10,000
638,4	0,000	1,678	3,479	5,478	7,677	10,000
672	0,000	1,709	3,528	5,528	7,708	10,000
705,6	0,000	1,737	3,573	5,573	7,736	10,000
739,2	0,000	1,762	3,614	5,614	7,761	10,000
772,8	0,000	1,784	3,651	5,651	7,784	10,000
806,4	0,000	1,805	3,684	5,684	7,805	10,000

METODOS IMPLICITOS

Crank-Nicolson

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\frac{u_{i+1}^{j} - 2u_{i}^{j} + u_{i-1}^{j}}{(\Delta x)^2} + \frac{u_{i+1}^{j+1} - 2u_{i}^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} \right)$$

$$\frac{\partial u}{\partial t} = \frac{u_{i}^{j+1} - u_{i}^{j}}{\Delta t}$$

Si: $$\frac{\Delta t}{\alpha(\Delta x)^2} = r$$

$$-ru_{i-1}^{j+1} + (2+2r)u_i^{j+1} - ru_{i+1}^{j+1} =$$
$$ru_{i-1}^{j} + (2-2r)u_i^{j} + ru_{i+1}^{j}$$

Si r=1:

$$-u_{i-1}^{j+1} + 4u_i^{j+1} - u_{i+1}^{j+1} = u_{i-1}^{j} + u_{i+1}^{j}$$

Fórmula implícita